Discrete Uniformization
and
Ideal Hyperbolic Polyhedra

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Discrete conformal maps and discrete uniformization.
\[ \tilde{l}_{ij} = e^{\frac{1}{2} (u_i + u_j)} l_{ij} = e^{\frac{u_i}{4}} e^{\frac{u_j}{4}} l_{ij} = \sqrt{e^{u_i} e^{u_j}} l_{ij} \] (Luo)
[Springborn, Schroder, Pinkall. Conformal Equivalence of Triangle Meshes]
Discrete conformal equivalence v.1

Two combinatorially equivalent triangle meshes \((T, \ell), (\tilde{T}, \tilde{\ell})\) are discretely conformally equivalent if there is a holomorphic
\[ u : V \to \mathbb{R} \]
such that
\[ \tilde{\ell}_{ij} = e^{\frac{i}{2} (uv + u'v')} \ell_{ij} \]
T - abstract triangulation.

l : E → IR edge length function.
Discrete conformal mapping problem

Given:
- a triangle mesh $(T, \ell)$
- desired angle sums $\Theta$

$\Theta : V \rightarrow IR_{\geq 0}$
(satisfying Gauss-Bonnet-condition)

Find a discrete conformal equ. mesh $(\tilde{T}, \tilde{\ell})$ with the given angle sums $\Theta$.

This means: Find $u : V \rightarrow IR$. 
Variational principle

\[ E_{T, \ell, \Theta} : \mathbb{R}^V \rightarrow \mathbb{R} \]

\[ u \sqcup E_{T, \ell, \Theta}(u) \]

Solutions of mapping problem \( \Rightarrow \) \( \text{grad } E_{T, \ell, \Theta} = 0 \).

\( \text{grad } E_{T, \ell, \Theta}(u) = 0 \) \( \Rightarrow \) \( \begin{cases} \text{\ell satisfies all triangle req. } \rightarrow \text{solution exists} \\ \text{triangle inequalities violated } \rightarrow \text{no solution exists} \end{cases} \)
Uniqueness: Yes. Existence: No.

\[ E_{T,e,0}(u) \text{ is convex! } \]

To solve the discr. unlocal happy problem, minimize \[ E_{T,e,0}(u) \].

\[ \rightarrow \text{ Solution of happy problem is unique up to scale. } \]

(If it exists.)
The hyperbolic metric induced by circumcircles

Euclidean triangle and circumcircle

Hyperbolic plane in Beltrami–Klein model and an ideal triangle
Discrete conformal equivalence & hyperbolic geometry

Essential observation:

Two combinatorially equivalent triangle meshes are discretely conformally equivalent if and only if they are isometric with respect to the induced hyperbolic metric.

[Bobenko, Pinkall, 5]
Two piecewise flat metrics $d, \tilde{d}$ on $(S,V)$ are discretely conformally equivalent if the hyperbolic metrics induced by the Delaunay triangulations of $(S,V,d)$ and $(S,V,\tilde{d})$ are isometric.
Equivalently:

Two Delaunay triangle meshes \((T, \ell), (\tilde{T}, \tilde{\ell})\) are discretely conformally equivalent if they are related by a combination of vertex scaling,

\[ l_{ij} \rightarrow \hat{l}_{ij} = e^{\frac{i}{2}(u_i + u_j)} l_{ij}, \]

and Ptolemy-flips:
\[ ef = ac + bd \]  \quad \text{(Ptolemy relation)}

Equivalent: Definition of Gu, Luo, Sun, Wo.
Ptolemy Flips...

- change shapes of triangles unless they share circumscribed circle

\[ ef = ac + bd \]

- do not change the induced hyperbolic metric
- commute with vertex scaling
Existence & uniqueness!

Given:

- closed piecewise flat surface \((S, \nu, d)\)
- desired cone angles \(\Theta: V \to \mathbb{R}_{>0}, \sum_v (2\pi - \Theta_v) = 2\pi \)

There exists a unique piecewise flat metric \(\tilde{d}\) on \((S, \nu)\) that is:

- discretely conformally equivalent and
- has cone angles \(\Theta\).

[Gu, Luo, Sun, Wu]
• For $\Theta = 2\pi$, genus = 1:
  Discrete Uniformization Theorem for Tori

• For higher genera:
  Analogous theory for piecewise hyperbolic surfaces.
  [Bobenko, Pinkall, S]
  [Gu, Guo, Luo, Sun, Wu] → Uniformization Theorems
  [Prosahov]
Theorem. For every piecewise flat metric on $\left( S^2, \mathcal{U} \right)$, there is a convex polyhedron inscribed in $S^2$ that is discretely conformally equivalent. It is unique up to a Möbius transformation applied to the vertices.
[Crewe, Gillespie, S] (in prep.)
Variational principle extends.
2.

Ideal hyperbolic polyhedra
Discrete Uniformization Theorems are equivalent to Realizability Theorems for Ideal Hyperbolic Polyhedra (with prescribed intrinsic metric)
Theorem (Rivièr). For every complete hyperbolic metric of finite area on $S^2 \setminus V$, there is a unique convex ideal polyhedron in $H^3$ realising the metric.

For tori & higher genus: Schlenker, Fillastre